JOURNAL OF APPROXIMATION THEORY 80, 253–266 (1995)

An Asymptotic Formula in Best Approximation*

NOLI N. REYES

Mathematics Department, The Ohio State University, Columbus, Ohio 43210

Communicated by Edward B. Saff

Received June 12, 1992; accepted in revised form August 2, 1994

We generalize some results of Saff and Varga on weighted approximation of a fixed monomial by a given finite-dimensional space of incomplete polynomials.

1. INTRODUCTION

The results presented in this paper were motivated by the work of Saff and Varga on incomplete polynomials. In [7], they obtained an asymptotic formula for the error in approximating a fixed monomial x^{μ} by a fixed l-dimensional space of incomplete polynomials of the form $\sum_{i=1}^{l} A_i x^{\mu_i}$ with respect to a weight $w_k(x) = x^k$, with $k \to \infty$. Here, $\mu_1, ..., \mu_l, \mu$ are fixed positive integers such that $\mu_1 < \cdots < \mu_l < \mu$. In more precise terms, they proved for $1 \leqslant p \leqslant \infty$,

$$\lim_{k \to \infty} k^{l+1/p} \inf_{A_i} \left\| x^k \left(x^{\mu} - \sum_{i=1}^{l} A_i x^{\mu_i} \right) \right\|_{L^p[0,1]} = \frac{e_p}{l!} \prod_{j=1}^{l} (\mu - \mu_j), \tag{1}$$

where

$$e_p := \inf_{P \in \pi_{l-1}} \| e^{-t} (t^l - P(t)) \|_{L^p[0,\infty)}, \tag{2}$$

and π_{l-1} denotes the set of all polynomials of degree at most $l-1$.

To rewrite (1) from a different perspective, let us define for $k = 1, 2, ...,$ and for $1 \leqslant p \leqslant \infty$,

$$z_i(k) = \frac{\mu_i - \mu}{\mu + k + 1/p}.$$

* Part of the author's Ph.D. thesis written under the supervision of Professor B. Baishanski.

0021-9045/95 $6.00

640 80 2-9

By a change of variable, $e^{-t} = x^{\mu + k + 1/p}$, and by letting $\phi(z, t) = e^{-t(1+z)}$, we may express (1) equivalently as follows:

$$\lim_{k \to \infty} \frac{\inf_{A_i} \|\phi(0, t) - \sum_{i=1}^{l} A_i \phi(z_i(k), t)\|_{L^p[0, \infty)}}{|z_1(k) \cdots z_l(k)|} = \frac{e_p}{l!}.$$

Given an arbitrary function ϕ of two variables satisfying reasonable conditions, this naturally raises the problem of determining the asymptotic behavior of the error in approximating the function $\phi(0, \cdot)$ by linear combinations of the l translates $\phi(z_i, \cdot)$, $i = 1, ..., l$, as $(z_1, ..., z_l) \to (0, ..., 0)$. To be more precise, if ϕ is defined on a compact rectangle in R^2 of the form $[-\rho, \rho] \times [a, b]$, what is the asymptotic behavior of

$$E_p(z) := \inf_{A_i} \left\| \phi(0, t) - \sum_{i=1}^{l} A_i \phi(z_i, t) \right\|_{L^p[a, b]} \tag{3}$$

as $z := (z_1, ..., z_l) \to (0, ..., 0)$? Not only shall we provide an answer to this. Our main result, as a matter of fact, will describe the asymptotic behavior of the extremal functions at each point $t \in [a, b]$:

THEOREM 1. *Let ϕ be defined on a compact rectangle in R^2 of the form $[-\rho, \rho] \times [a, b]$ such that*

(i) *for some constant $K > 0$,*

$$|\det \phi(z_i, t_j)| \geqslant K \prod_{r < s} |(z_s - z_r)(t_s - t_r)|, \tag{4}$$

whenever $z_1, ..., z_l \in [-\rho, \rho]$ and $t_1, ..., t_l \in [a, b]$, and

(ii) *for $i = 0, ..., l+1$, and $j = 0, ..., l-1$,*

$$\frac{\partial^{i+j}\phi}{\partial z^i\, \partial t^j} \text{ is continuous on } [-\rho, \rho] \times [a, b]. \tag{5}$$

For $1 \leqslant p \leqslant \infty$ and for any l-tuple $z = (z_1, ..., z_l)$ with distinct nonzero entries $z_i \in [-\rho, \rho]$, define

$$P_{p,z}(t) := \phi(0, t) - \sum_{k=1}^{l} A_{k,p}(z)\, \phi(z_k, t)$$

and

$$P_p^*(t) := \frac{\partial^{(l)}\phi}{\partial z^{(l)}}(0, t) - \sum_{k=0}^{l-1} B_{k,p}^* \frac{\partial^{(k)}\phi}{\partial z^{(k)}}(0, t),$$

to be the unique such linear combinations that satisfy

$$\|P_{p,z}\|_{L^p[a,b]} = E_p(z) \quad \text{and} \quad \|P_p^*\|_{L^p[a,b]} = E_p^*$$

respectively, where $E_p(z)$ *is given in* (3) *and we define*

$$E_p^* := \inf_{B_0,\ldots,B_{l-1}} \left\| \frac{\partial^{(l)}\phi}{\partial z^{(l)}}(0,t) - \sum_{i=0}^{l-1} B_i \frac{\partial^{(i)}\phi}{\partial z^{(i)}}(0,t) \right\|_{L^p[a,b]}. \tag{6}$$

Then for $1 < p \leqslant \infty$,

$$\lim_{z \to (0,\ldots,0)} \frac{P_{p,z}(t)}{z_1 \cdots z_l} = \frac{(-1)^l}{l!} P_p^*(t), \quad \text{uniformly for} \quad t \in [a,b]. \tag{7}$$

If $p = 1$, (7) *also holds if in addition, we assume that any linear combination* $\phi(0,t) - \sum_{i=1}^{l} A_i \phi(z_i, t)$ *is nonzero for almost all* $t \in [a,b]$.

COROLLARY 1. *With* ϕ *given as in Theorem* 1,

$$\lim_{z \to (0,\ldots,0)} \frac{E_p(z)}{|z_1 \cdots z_l|} = \frac{E_p^*}{l!}, \qquad 1 < p \leqslant \infty, \tag{8}$$

where the limit is taken with the z_i*'s remaining distinct and non-zero. For* $p = 1$, (8) *also holds if, in addition, we assume that any linear combination* $\phi(0,t) - \sum_{i=1}^{l} A_i \phi(z_i, t)$ *is nonzero for almost all* $t \in [a,b]$.

Conditions (4) and (5) imply that the following two sets of l functions

$$\phi(z_k, \cdot), \qquad k = 1, \ldots, l \qquad (\text{with distinct } z_k\text{'s}) \tag{9}$$

$$\frac{\partial^{(k-1)}\phi}{\partial z^{(k-1)}}(0, \cdot), \qquad k = 1, \ldots, l, \tag{10}$$

are Chebyshev systems on $[a,b]$. This is trivial for (9). To prove the assertion for (10), we rewrite the inequality (4) by performing a series of elementary row transformations, obtaining:

$$\begin{vmatrix} [\phi]_0(t_1) \cdots [\phi]_{l-1}(t_1) \\ \cdots \\ [\phi]_0(t_l) \cdots [\phi]_{l-1}(t_l) \end{vmatrix} \geqslant K \prod_{r<s} |t_s - t_r|,$$

where $[\phi]_k(t)$ denotes the kth divided difference of $\phi(z,t)$ for $z = z_1, \ldots, z_{k+1}$. The assertion for (10) then follows by letting $z_1, \ldots, z_l$ tend to zero. Consequently, for $1 \leqslant p \leqslant \infty$, the best L^p-approximation of any continuous function on $[a,b]$ by linear combinations of (9) or (10) is unique. This is a classical result for $1 < p \leqslant \infty$. For $p = 1$, one may refer to [8, p. 38].

The main tool in the proof of Theorem 1 is a mean value theorem, of interest by itself, presented in the next section.

Finally, using our results and some finite–infinite range inequalities, we are able to generalize Saff and Varga's asymptotic formula (1) in the following form:

THEOREM 2. *For each positive integer k, let $\mu(k), \mu_1(k), ..., \mu_l(k)$ be $l+1$ distinct real numbers tending to infinity as $k \to \infty$ such that*

$$\lim_{k \to \infty} \frac{\mu_i(k)}{\mu(k)} = 1, \qquad i = 1, ..., l.$$

Then for $1 \leqslant p \leqslant \infty$, and with e_p defined in (2),

$$\lim_{k \to \infty} \frac{\mu(k)^{1/p+l} \inf_{A_i} \|x^{\mu(k)} - \sum_{i=1}^{l} A_i x^{\mu_i(k)}\|_{L^p[0,1]}}{\prod_{i=1}^{l} |\mu_i(k) - \mu(k)|} = \frac{e_p}{l!}. \tag{11}$$

The proofs are given in Sections 3, 4, and 5.

2. A Mean Value Theorem and Some Examples

In this section, ϕ will be defined on $[-\rho, \rho] \times I$, where ρ is a given positive real number and I is an interval of the real line, possibly unbounded. Moreover, it will always be assumed that the partial derivatives

$$\frac{\partial^{i+j}\phi}{\partial z^i \, \partial t^j}$$

are bounded on $[-\rho, \rho] \times I$ for $i = 0, ..., l+1$, and $j = 0, ..., l-1$.

THEOREM 3. *Let J be a sub-interval of I (possibly the whole of I), for which there is a constant K satisfying*

$$|\det \phi(z_i, t_j)| \geqslant K \prod_{r<s} |(z_s - z_r)(t_s - t_r)|, \tag{12}$$

whenever $z_1, ..., z_l \in [-\rho, \rho]$ and $t_1, ..., t_l \in J$. Then there exist positive constants M (depending only on ϕ, I, ρ, and l) and δ (depending only on ϕ, I, J, ρ, and l) such that whenever

(1) $\max\{|z_1|, ..., |z_l|\} < \delta$ *and*

(2) $P(t) := \phi(0, t) - \sum_{i=1}^{l} A_i \phi(z_i, t)$ *has* l *distinct zeros on* J, *then for* $t \in I$

$$P(t) = \frac{(-1)^l}{l!} z_1 \cdots z_l \left(\frac{\partial^{(l)}\phi}{\partial z^{(l)}}(0, t) - \sum_{k=0}^{l-1} B_k \frac{\partial^{(k)}\phi}{\partial z^{(k)}}(0, t) + R(t) \right), \quad (13)$$

where $|B_k| \leqslant MK^{-1}$, $k = 0, 1, ..., l-1$, *and* $|R(t)| \leqslant MK^{-2} \max |z_k|$, *for all* $t \in [a, b]$.

We note that if J is a bounded interval, condition (12) is satisfied by the function $\phi(z, t) = e^{-t(1+z)}$. See [4, p. 15]. In fact, the so-called extended sign-regular functions ϕ on $[-\rho, \rho] \times J$ treated extensively in [4] would also satisfy condition (12). These are functions ϕ such that all its partial derivatives of order $2l+2$ are continuous and for which there is a sequence $\varepsilon_0, \varepsilon_1, ..., \varepsilon_{l+1}$ (where $\varepsilon_r = +1$ or -1), satisfying

$$\varepsilon_r \begin{vmatrix} \phi(z, t) & \cdots & \dfrac{\partial^r \phi}{\partial z^r}(z, t) \\ & \cdots & \\ \dfrac{\partial^r \phi}{\partial t^r}(z, t) & \cdots & \dfrac{\partial^{2r}\phi}{\partial z^r \, \partial t^r}(z, t) \end{vmatrix} > 0, \quad (14)$$

for any $(z, t) \in [-\rho, \rho] \times J$, and for $r = 0, 1, ..., l+1$.

Karlin, in [4], gives various examples of functions ϕ extended sign-regular on a rectangle $X \times Y$ in R^2. For example, it is shown that given a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ $(a_n > 0)$, with a positive radius of convergence r, the function ϕ defined by $\phi(z, t) := f(t(1+z))$ is extended sign-regular on any rectangle lying on the region $\{(z, t) \in R^2 : z \geqslant -1, t \geqslant 0, t(1+z) \leqslant r\}$. See [4, p. 101].

3. Proof of Theorem 3

Given l points $z_1, ..., z_l$ on $[-\rho, \rho]$ and l numbers $t_1, ..., t_l$ on J, we shall adopt the following notation:

(i) $[\phi]_k(t)$ for the kth divided difference of $\phi(z, t)$ for $z = z_0, z_1, ..., z_k$, $k = 0, ..., l$ where $z_0 = 0$.

(ii) $[\phi]_{k,m-1}$ for the $m-1$st divided difference of $[\phi]_k(t)$ for $t = t_1, ..., t_m$, $m = 1, ..., l$.

(iii) C_0 for the quantity

$$\sup_{i,j} \frac{1}{i!\, j!} \left\| \frac{\partial^{i+j}\phi}{\partial z^i \, \partial t^j} \right\|,$$

the supremum being taken over $0 \leqslant i \leqslant l+1$, $0 \leqslant j \leqslant l-1$, and $\|\cdot\|$ denotes the supremum norm over $[-\rho, \rho] \times I$.

For convenience, we introduce here once and for all, notation for the determinants that will appear in the proof:

$$D := \begin{vmatrix} [\phi]_{0,0} & \cdots & [\phi]_{l-1,0} \\ & \cdots & \\ [\phi]_{0,l-1} & \cdots & [\phi]_{l-1,l-1} \end{vmatrix}, \quad D(t) := \begin{vmatrix} \phi(0,t) & \cdots & \dfrac{1}{l!}\dfrac{\partial^l \phi}{\partial z^l}(0,t) \\ [\phi]_{0,0} & \cdots & [\phi]_{l,0} \\ & \cdots & \\ [\phi]_{0,l-1} & \cdots & [\phi]_{l,l-1} \end{vmatrix},$$

$$\Delta = \begin{vmatrix} (-1)^l z_1 \cdots z_l & \cdots & -z_l & 1 \\ [\phi]_0(t_1) & \cdots & [\phi]_{l-1}(t_1) & [\phi]_l(t_1) \\ & \cdots & & \\ [\phi]_0(t_l) & \cdots & [\phi]_{l-1}(t_l) & [\phi]_l(t_l) \end{vmatrix},$$

$$\Delta_1 := \begin{vmatrix} (-1)^l z_1 \cdots z_l & \cdots & -z_l & 1 \\ [\phi]_{0,0} & \cdots & [\phi]_{l-1,0} & [\phi]_{l,0} \\ & \cdots & & \\ [\phi]_{0,l-1} & \cdots & [\phi]_{l-1,l-1} & [\phi]_{l,l-1} \end{vmatrix},$$

$$\Delta(t) = \begin{vmatrix} [\phi]_0(t) & \cdots & [\phi]_l(t) \\ [\phi]_0(t_1) & \cdots & [\phi]_l(t_1) \\ & \cdots & \\ [\phi]_0(t_l) & \cdots & [\phi]_l(t_l) \end{vmatrix}, \quad \Delta_1(t) := \begin{vmatrix} [\phi]_0(t) & \cdots & [\phi]_l(t) \\ [\phi]_{0,0} & \cdots & [\phi]_{l,0} \\ & \cdots & \\ [\phi]_{0,l-1} & \cdots & [\phi]_{l,l-1} \end{vmatrix}.$$

First of all, we claim that for some $\delta > 0$ (independent of the t_i's and of the z_j's)

$$|\Delta_1| \geqslant \frac{K}{2}, \tag{15}$$

for any choice of z_i's in $[-\delta, \delta]$ and t_j's in J. Indeed, by expanding this determinant with respect to the first row, we obtain

$$|D - (-1)^l \Delta_1| \leqslant l!\, C_0^l \sum_{k=1}^{l} |z_k \cdots z_l| \leqslant l!\, C_0^l \frac{1-\rho^l}{1-\rho} \max_j |z_j|, \tag{16}$$

whenever $z_1, ..., z_l \in [-\rho, \rho]$. Meanwhile, observe that condition (12) implies

$$|D| \geqslant K \tag{17}$$

for any choice of z_i's in $[-\rho, \rho]$ and t_j's in J. (This follows immediately by a series of row and column transformations.) Therefore if δ is chosen to be any positive number less than or equal to

$$\frac{KC_0^{-l}(1-\rho)}{2l!\,(1-\rho^l)},$$

(17) combined with (16) implies (15) provided that $\max_j |z_j| < \delta$.

Now, fix l distinct nonzero numbers $z_1, ..., z_l$ on $(-\delta, \delta)$ and assume that $P(t) := \phi(0, t) - \Sigma_{i=1}^{l} A_i \phi(z_i, t)$ has l distinct zeros $t_1 < \cdots < t_l$ on J. After some manipulations, one arrives at rewriting $P(t)$ as a linear combination of the divided differences $[\phi]_k(t)$:

$$P(t) := \sum_{k=0}^{l} \alpha_k [\phi]_k(t). \tag{18}$$

Collecting coefficients of $\phi(0, t)$, we obtain

$$\alpha_0 + \sum_{k=1}^{l} \frac{\alpha_k(-1)^k}{z_1 \cdots z_k} = 1. \tag{19}$$

Since $P(t)$ vanishes at $t_1, ..., t_l$, we also have

$$\sum_{k=0}^{l} \alpha_k [\phi]_k(t_j) = 0, \qquad j = 1, ..., l. \tag{20}$$

We remark that the determinant of the system (19)–(20) of $l+1$ equations is nonzero. Indeed, we may write that determinant as

$$\frac{(-1)^l \Delta}{z_1 \cdots z_l}. \tag{21}$$

By a series of row transformations, one obtains $\Delta = \Delta_1 \prod_{i<j}(t_j - t_i)$. The inequality in (15) then shows that the determinant of the system (19)–(20), which is given by (21), never vanishes provided the z_i's are nonzero, as we have assumed them to be.

Now, solving the system (19)–(20) by Kramer's Rule and substituting in (18), we obtain $P(t) = (-1)^l z_1 \cdots z_l \Delta(t) \Delta^{-1}$. By virtue of the identity $\Delta(t) := \Delta_1(t) \prod_{i<j}(t_j - t_i)$ (which can be obtained by a series of row transformations), we may write $P(t) = (-1)^l z_1 \cdots z_l \Delta_1(t) \Delta_1^{-1}$.

In the meantime, observe that the quotient $D(t)/D$, takes the form

$$\frac{D(t)}{D} := \frac{(-1)^l}{l!}\left(\frac{\partial^{(l)}\phi}{\partial z^{(l)}}(0, t) - \sum_{k=0}^{l-1} B_k \frac{\partial^{(k)}\phi}{\partial z^{(k)}}(0, t)\right) \tag{22}$$

and we may write

$$P(t) = (-1)^l z_1 \cdots z_l \left((-1)^l \frac{D(t)}{D} + R^*(t)\right),$$

where

$$R^*(t) := \frac{\Delta_1(t)}{\Delta_1} - \frac{(-1)^l D(t)}{D}.$$

So now, it remains for us to obtain estimates for the coefficients B_k, and the remainder $R^*(t)$. In view of (17) and (22), we easily obtain the following estimates for the coefficients B_k, $k = 0, ..., l-1$

$$|B_k| \leqslant \frac{M_k}{K}, \tag{23}$$

where

$$M_k := \frac{(l!)^2 C_0^l}{k!}, \qquad k = 0, ..., l-1. \tag{24}$$

To see this, observe that

$$\frac{\partial^{(k)}\phi}{\partial z^{(k)}}(0, t)$$

appears in $D(t)$, having as coefficient $(k!)^{-1}(-1)^k$ multiplied by an $l \times l$ determinant whose entries are the divided differences $[\phi]_{i,j}$ ($0 \leqslant i \leqslant l$, $i \neq k$, $0 \leqslant j \leqslant l-1$), each of which is majorized by C_0.

To estimate the remainder $R^*(t)$, we note that (17) and (15) imply

$$|\Delta_1 \cdot D| \geqslant \frac{K^2}{2}. \tag{25}$$

Moreover,

$$|D| \leqslant l!\, C_0^l \qquad \text{and} \qquad |D(t)| \leqslant (l+1)!\, C_0^{l+1}, \qquad t \in I. \tag{26}$$

Furthermore, for each $j=0, \dots, l$, we obtain numbers ζ_j, ζ_j^*, satisfying $|\zeta_j|, |\zeta_j^*| < \max\{|z_1|, \dots, |z_l|\}$ such that

$$[\phi]_j(t) - \frac{1}{j!}\frac{\partial^j \phi}{\partial z^j}(0, t) = \frac{\zeta_j^*}{j!}\frac{\partial^{j+1}\phi}{\partial z^{j+1}}(\zeta_j, t).$$

This implies that for $t \in I$ and for $j=0, \dots, l$

$$\left|[\phi]_j(t) - \frac{1}{j!}\frac{\partial^j \phi}{\partial z^j}(0, t)\right| \leqslant (j+1)\, C_0 \cdot \max_k |z_k|,$$

which brings us to the following estimate, valid for all $t \in [a, b]$:

$$|\Delta_1(t) - D(t)| \leqslant (l+2)!\, C_0^{l+1} \max_k |z_k|. \tag{27}$$

Finally, combining (16), (25), (26), and (27) we obtain

$$|R^*(t)| \leqslant \frac{M_l}{l!\, K^2} \max_k |z_k|, \qquad t \in I,$$

where

$$M_l := 2((l+1)!)^3\, C_0^{2l+1}\left(l + 2 + \frac{1-\rho^l}{1-\rho}\right).$$

Therefore, by taking $R(t) := l!\, R^*(t)$, and $M := \max\{M_0, M_1, \dots, M_l\}$, where $M_0, M_1, \dots, M_{l-1}$ have been defined in (24), we complete the proof of Theorem 3. Q.E.D.

4. Proof of Theorem 1

For each p, $1 \leqslant p \leqslant \infty$, and for each l-tuple $z = (z_1, \dots, z_l)$ with distinct nonzero entries z_j in $[-\rho, \rho]$, $P_{p,z}$ will have l distinct zeros on $[a, b]$. See for example [9, p. 98]. This allows us to apply Theorem 3 with $I = J = [a, b]$. So there exists $\delta > 0$ such that whenever $\max_k |z_k| < \delta$,

$$P_{p,z}(t) = \frac{(-1)^l}{l!} z_1 \cdots z_l \left(\frac{\partial^{(l)}\phi}{\partial z^{(l)}}(0, t) - \sum_{k=0}^{l-1} B_{k,p}(z) \times \frac{\partial^{(k)}\phi}{\partial z^{(k)}}(0, t) + R_p(z, t)\right), \tag{28}$$

for $t \in [a, b]$ where

$$|B_{k,p}(z)| \leqslant C, \qquad k = 0, ..., l-1, \tag{29}$$

and

$$|R_p(z, t)| \leqslant C \cdot \max_k |z_k|, \qquad t \in [a, b], \tag{30}$$

for some constant C depending only on ϕ, a, b, ρ, p, and l.

Thus, to prove the theorem, it will be sufficient to show that

$$\lim_{z \to (0, ..., 0)} B_{k,p}(z) = B^*_{k,p}, \qquad \text{for} \quad k = 0, ..., l-1.$$

Assuming the contrary, there exists $\varepsilon > 0$, such that

$$|B_{k,p}(z^{(j)}) - B^*_{k,p}| \geqslant \varepsilon, \qquad j = 1, 2, 3, ..., \tag{31}$$

for some k, $0 \leqslant k \leqslant l-1$, and for some sequence of l-tuples $\{z^{(j)}\}_{j=1}^{\infty}$ tending to $(0, ..., 0)$.

(29) implies the existence of a subsequence of $\{z^{(j)}\}_{j=1}^{\infty}$, which we shall denote again by $\{z^{(j)}\}_{j=1}^{\infty}$, for which

$$\tilde{B}_{k,p} := \lim_{j \to \infty} B_{k,p}(z^{(j)}) \text{ exists}$$

for $k = 0, ..., l-1$. By defining

$$\tilde{P}_p(t) := \frac{\partial^{(l)}\phi}{\partial z^{(l)}}(0, t) - \sum_{k=0}^{l-1} \tilde{B}_{k,p} \frac{\partial^{(k)}\phi}{\partial z^{(k)}}(0, t),$$

we obtain from (28):

$$\lim_{j \to \infty} \frac{P_{p, z^{(j)}}(t)}{z_1^{(j)} \cdots z_l^{(j)}} = \frac{(-1)^l}{l!} \tilde{P}_p(t), \qquad \text{uniformly for} \quad t \in [a, b]. \tag{32}$$

We claim that for $1 \leqslant p \leqslant \infty$,

$$\tilde{P}_p(t) = P^*_p(t), \qquad t \in [a, b]. \tag{33}$$

(This would contradict (31), thus completing the proof of Theorem 1.)

Proof of (33) *for* $p = \infty$. If $\tilde{P}_\infty \equiv 0$ on $[a, b]$, then (33) immediately follows by uniqueness of the best L^∞-approximation by Chebyshev systems. So we may assume that $\|\tilde{P}_\infty\|_{L^\infty[a,b]} > 0$. Being the uniform limit of a sequence of functions each with $l+1$ equioscillation points on $[a, b]$, $\tilde{P}_\infty(t)$ itself must also have $l+1$ equioscillation points on $[a, b]$. See

[8, p. 56]. Thus, $\|\tilde{P}_\infty\|_{L^\infty[a,b]} = E^*_\infty$, and from this, (33) follows for $p=\infty$, by uniqueness of the best L^∞-approximation by Chebyshev systems.

Proof of (33) *for* $1 \leqslant p < \infty$. The characterization of the best L^p-approximation [8, p. 64], implies

$$\int_a^b \phi(z_k, t)\, |P_{p,z}(t)|^{p-1} \operatorname{sgn} P_{p,z}(t)\, dt = 0, \qquad k = 1, ..., l.$$

(Recall that for $p=1$, our assumptions imply that $P_{p,z}(t) \neq 0$ for almost all $t \in [a, b]$.) For $k=0, ..., l-1$, let $(\phi)_k(t)$ denote the kth divided difference of $\phi(z, t)$ for $z = z_1, ..., z_{k+1}$. Since $z_1, ..., z_l$ are distinct and nonzero, $(\phi)_k(t)$ is a linear combination of $\phi(z_1, t), ..., \phi(z_{k+1}, t)$. Hence

$$\int_a^b (\phi)_k(t) \left| \frac{P_{p,z}(t)}{z_1 \cdots z_l} \right|^{p-1} \operatorname{sgn} \frac{P_{p,z}(t)}{z_1 \cdots z_l}\, dt = 0, \qquad k = 0, ..., l-1.$$

Now, we let $z = (z_1, ..., z_l) \to (0, ..., 0)$ through the sequence $\{z^{(j)}\}_{j=0}^\infty$. Applying (32) and the Lebesgue Dominated Convergence Theorem, we obtain

$$\int_a^b \frac{\partial^k \phi}{\partial \zeta^k}(0, t)\, |\tilde{P}_p(t)|^{p-1} \operatorname{sgn} \tilde{P}_p(t)\, dt = 0, \qquad k = 0, ..., l-1. \tag{34}$$

This shows that $\|\tilde{P}_p\|_{L^p[a,b]} = E_p^*$, for $1 \leqslant p < \infty$. Uniqueness of the best L^p-approximation by Chebyshev systems then implies that (33) also holds for $1 \leqslant p < \infty$. Q.E.D.

5. Proof of Theorem 2

We shall be applying the following finite–infinite range inequalities. Mentioned here are only very special cases of the results of Mhaskar, Saff, and Varga from [5] and [6].

(1) For each $p > 0$, there exist positive constants c_1, c_2, depending only on p such that for each integer $\lambda \geqslant 1$, and for each polynomial $Q \in \pi_\lambda$,

$$\int_0^\infty |e^{-t} Q(t)|^p\, dt \leqslant (1 + c_1 \exp(-c_2 \lambda))^p \int_0^{3\lambda} |e^{-t} Q(t)|^p\, dt.$$

(2) For any polynomial P of degree at most λ,

$$\|e^{-t} P(t)\|_{L^\infty[0,\infty)} = \|e^{-t} P(t)\|_{L^\infty[0,2\lambda]}.$$

As an immediate consequence of these two results, we have for $1 \leqslant p \leqslant \infty$,

$$e_p = \lim_{\lambda \to \infty} e_p(\lambda) \tag{35}$$

where e_p is given in (2) and $e_p(\lambda)$ is defined by

$$e_p(\lambda) := \inf_{P \in \pi_{l-1}} \|e^{-t}(t^l - P(t))\|_{L^p[0, 3\lambda)}. \tag{36}$$

In what follows, p will be fixed such that $1 \leqslant p \leqslant \infty$. By defining

$$z_i(k) = \frac{\mu_i(k) - \mu(k)}{\mu(k) + 1/p}$$

and employing the change of variable $e^{-t} = x^{\mu(k) + 1/p}$ we may rewrite (11) equivalently as follows

$$\lim_{k \to \infty} \frac{\hat{E}_p(k)}{|z_1(k) \cdots z_l(k)|} = \frac{e_p}{l!}, \tag{37}$$

where

$$\hat{E}_p(k) := \inf_{A_i} \left\| e^{-t} - \sum_{i=1}^{l} A_i e^{-t(1 + z_i(k))} \right\|_{L^p[0, \infty)}. \tag{38}$$

Note that by taking $\phi(z, t) = e^{-t(1+z)}$ in Theorem 3, we easily obtain a finite-interval version of (37). Namely, for each $\lambda > 0$,

$$\lim_{k \to \infty} \frac{\inf_{A_i} \|e^{-t} - \sum_{i=1}^{l} A_i e^{-t(1 + z_i(k))}\|_{L^p[0, 3\lambda]}}{|z_1(k) \cdots z_l(k)|} = \frac{e_p(\lambda)}{l!},$$

where $e_p(\lambda)$ is defined in (36). Equation (35) then implies that

$$\liminf_{k \to \infty} \frac{\hat{E}_p(k)}{|z_1(k) \cdots z_l(k)|} \geqslant \frac{e_p}{l!}. \tag{39}$$

To prove the inequality

$$\limsup_{k \to \infty} \frac{\hat{E}_p(k)}{|z_1(k) \cdots z_l(k)|} \leqslant \frac{e_p}{l!}, \tag{40}$$

we proceed with an indirect argument. Assuming the contrary, we can find an increasing sequence $k_1 < k_2 < \cdots$ of positive integers, and a positive number ε such that

$$\frac{\hat{E}_p(k_j)}{|z_1(k_j) \cdots z_l(k_j)|} > \frac{e_p}{l!} + \varepsilon, \qquad j = 1, 2, ... \tag{41}$$

Now, define real numbers $A_{i,j}$, $1 \leqslant i \leqslant l$, $j = 1, 2, \ldots$ such that the functions defined by

$$Q_j(t) := e^{-t} - \sum_{i=1}^{l} A_{i,j} e^{-t(1+z_i(k_j))}, \qquad j = 1, 2, 3, \ldots,$$

vanishes precisely at the zeros of $t^l - P^*(t)$ where $P^*(t) \in \pi_{l-1}$ is the unique polynomial satisfying $\|e^{-t}(t^l - P^*(t))\|_{L^p[0,\infty)} = e_p$.

By taking $\phi(z, t) = e^{-t(1+z)}$, $I = [0, \infty)$, and J to be any fixed compact interval containing the zeros of $t^l - P^*(t)$, Theorem 3 asserts that there are polynomials $P_j \in \pi_{l-1}$, $j \geqslant 1$, such that for $0 \leqslant t < \infty$

$$Q_j(t) = \frac{(-1)^l}{l!} z_1(k_j) \cdots z_l(k_j)(e^{-t}(t^l - P_j(t)) + R_j(t)). \tag{42}$$

Moreover, the absolute values of the coefficients of the P_j's are less than some constant M (depending only on p), and as well, for $j = 1, 2, \ldots$, and $t \in [0, \infty)$, $|R_j(t)| \leqslant M \max\{|z_1(k_j)|, |z_2(k_j)|, \ldots, |z_l(k_j)|\}$. Since the right hand side of (42) vanishes precisely at the zeros of $t^l - P^*(t)$, we can find an increasing sequence $j_1 < j_2 < \cdots$, of positive integers such that the coefficients of $P_{j_n}(t)$ converge respectively to those of $P^*(t)$ as $n \to \infty$. Consequently, for $1 \leqslant p \leqslant \infty$,

$$\lim_{n \to \infty} \|e^{-t}(t^l - P_{j_n}(t))\|_{L^p[0,\infty)} = \|e^{-t}(t^l - P^*(t))\|_{L^p[0,\infty)} = e_p.$$

Therefore, taking the L^p-norm on $[0, \infty)$ of both sides of (42), and applying the triangle inequality on the right hand side of the resulting equation, lead us to the inequality

$$\limsup_j \frac{\hat{E}_p(k_j)}{|z_1(k_j) \cdots z_l(k_j)|} \leqslant \frac{e_p}{l!},$$

where the lim sup is taken as j runs over the sequence $j_1 < j_2 < \cdots$. This contradicts (41), thus establishing the validity of (40). This completes the proof of Theorem 2. Q.E.D.

References

1. B. Baishanski, Given two spaces of generalized Dirichlet polynomials, which one is closer to x^c? *in* "Proc. Internat. Cont. Constr. Function Theory Varna," *Bulgar. Acad. Sci.* (1984), 145–149.
2. B. Baishanski and R. Bojanic, An estimate for the coefficients of polynomials of given length, *J. Approx. Theory* **24** (1986), 181–188.
3. A. O. Gelfond, "Calcul des Différences Finies," Dunod, Paris, 1963.

4. S. Karlin, "Total Positivity," Vol. 1, Stanford Univ. Press, Stanford, CA, 1968.
5. E. B. Saff and R. S. Varga, On incomplete polynomials, II, *Pacific J. Math.* **92** (1981), 161–172.
6. H. N. Mhaskar and E. B. Saff, Where does the L^p-norm of a weighted polynomial live? *Trans. Amer. Math. Soc.* **303** (1987), 109–124; Erratum, *Trans. Amer. Math. Soc.* **308** (1988), 431.
7. E. B. Saff and R. S. Varga, On lacunary incomplete polynomials, *Math. Z.* **177** (1981), 297–314.
8. A. F. Timan, "Theory of Approximation of Functions of a Real Variable," MacMillan, New York, 1963.
9. G. A. Watson, "Approximation Theory and Numerical Methods," Wiley, New York, 1980.